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1997 J. Phys. A: Math. Gen. 30 661

(http://iopscience.iop.org/0305-4470/30/2/026)

J. Phys. A: Math. Gen. **30** (1997) 661–671. Printed in the UK PII: S0305-4470(97)76575-2

# Obtaining holonomy from curvature

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Received 16 July 1996

**Abstract.** In physical applications of differential geometry, one sometimes wishes to compute the holonomy group of a Riemannian manifold from local data, such as the curvature tensor. In general, this can be a complicated problem, but we show that, in cases of most interest in physics, the holonomy group can be obtained directly from the Lie algebras generated by the curvature tensor.

## 1. Introduction

In physical applications of Riemannian geometry, the holonomy group [1, 2] is frequently of basic interest [3–5]. Often the physical situation gives us some information on the curvature tensor, and we wish to use this to determine or at least constrain the holonomy group. In general, however, this is no easy task, and several interesting pathologies are possible.

Consider the work of Gibbons *et al* [4] which is concerned with the construction of certain manifolds $M$ of special holonomy. (Attention has recently been focused on such manifolds in connection with supermembrane theory [5].) In that work, the Lie algebra $\mathfrak{K}_x$ generated by the curvature tensor at $x \in M$ is computed. Gibbons *et al* are careful to emphasize that *it is not necessarily the case* that $\mathfrak{K}_x$ is isomorphic to the Lie algebra of the holonomy group; they remark that information on the covariant derivatives of the curvature tensor would be needed to bridge the gap between curvature and holonomy. In fact, there are also cases in which even this additional information would not be sufficient, even if it were available—which is rarely the case.

The overall purpose of this work is to investigate the relationship between the curvature tensor and the holonomy algebra of a Riemannian manifold, henceforth denoted $M$. In particular, we wish to establish results which allow us to deduce the algebra of the holonomy group from information on the curvature algebras, $\{\mathfrak{K}_x\}$. (The holonomy group itself is not completely determined by its algebra, because the group is often disconnected; but this problem is now rather well understood. Essentially, the additional information required is topological, involving the fundamental group of $M$; one compares this with the holonomy classification theorems. The full holonomy group can always be deduced in this way. See [6] for examples and further references.) Our most useful result in this direction is the following.

Let $x \in M$, and let $R_x$ be the curvature tensor at $x$. If $M$ is a local Riemannian product, then the tangent space $T_x(M)$ will split into a direct sum of subspaces $T_x^{(i)}$ of dimensions $m_i$, $\Sigma m_i = n = \dim M$, corresponding to a splitting of $\mathfrak{K}_x$ into a direct sum of sub-algebras,

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$\mathfrak{K}_x^{(i)}$. We shall say that $\mathfrak{K}_x$ is *generic* if any of the $\mathfrak{K}_x^{(i)}$ is isomorphic either to the orthogonal algebra $\mathfrak{SO}(m_i)$ or to the unitary algebra $\mathfrak{U}(m_i/2)$, if $m_i$ is even. (If $\mathfrak{K}_x$ does not split, 'generic' just means 'isomorphic to $\mathfrak{SO}(n)$ or $\mathfrak{U}(n/2)$', which are the holonomy algebras of *generic* [1] Riemannian and Kaehlerian manifolds, respectively.) Then we have the following result.

*Theorem 1.1.* Let $M$ be a complete, connected Riemannian manifold, and let $\mathfrak{G}$ be a sub-algebra of $\mathfrak{SO}(n)$ such that the curvature algebras satisfy

$$\mathfrak{K}_y \subseteq \mathfrak{G} \qquad \text{for all } y \in M$$
$$\mathfrak{K}_x = \mathfrak{G} \qquad \text{for some } x \in M.$$

(That is, all of the curvature algebras are contained in a certain 'minimal' one.) Then if $\mathfrak{K}_x$ is not generic, the Lie algebra of the holonomy group of $M$ is isomorphic to $\mathfrak{G}$.

The proof will be given in section 4, below.

This theorem immediately settles the concerns of Gibbons *et al* [4], for whom $\mathfrak{G}$ is either the $\mathfrak{G}_2$ sub-algebra of $\mathfrak{SO}(7)$ or the $\mathfrak{Spin}(7)$ sub-algebra of $\mathfrak{SO}(8)$, neither of which is generic of course. In practice, theorem 1.1 means that the holonomy algebra can be computed directly from the curvature algebras, provided that the latter are never generic—which is precisely the case of interest in physics.

The proof of such results combines results on the analyticity of Einstein manifolds [7] with the classical theorems of Nijenhuis [8, 9]. We begin with the basic machinery (see also [10]).

## 2. Infinitesimal holonomy

Let $M$ be a connected $n$-dimensional Riemannian manifold with metric $g$, Levi-Civitá connection $\nabla$, and curvature tensor $R$. Note that since $R$ is a $(1, 3)$ tensor, then for each pair of tangent vectors $X, Y$ at a point $x$, $R_x(X, Y)$ is a $(1, 1)$ tensor; in other words, it is just a linear map from the tangent space to itself, a so-called endomorphism of the tangent space. Let $\mathrm{End}[T_x(M)]$ be the algebra of endomorphisms of the tangent space $T_x(M)$; then $\mathfrak{K}_x$, the curvature algebra at $x$, is the sub-algebra of $\mathfrak{SO}(n)$ generated by the subspace of End $[T_x(M)]$ spanned by all $T_x(M)$ endomorphisms of the form $R_x(X, Y)$, where $X, Y \in T_x(M)$. Similarly, let $\mathfrak{K}_x^{(1)}$ be generated by all endomorphisms of the form $R_x(X, Y)$ or of the form $(\nabla R)_x(X, Y, Z)$, and so on. We have

$$\mathfrak{K}_x = \mathfrak{K}_x^{(0)} \subseteq \mathfrak{K}_x^{(1)} \subseteq \mathfrak{K}_x^{(2)} \subseteq \cdots$$

and we define the *infinitesimal holonomy algebra* $\mathfrak{T}_x$ by

$$\mathfrak{T}_x = \bigcup_{m=0}^{\infty} \mathfrak{K}_x^{(m)}$$

(see [10]). If $\mathfrak{Hol}_x(M)$ is the Lie algebra of the holonomy group $\mathrm{Hol}_x(M)$ at $x$, then

$$\mathfrak{T}_x \subseteq \mathfrak{Hol}_x(M) \qquad \text{for all } x \in M.$$

An interesting object which interpolates between $\mathfrak{T}_x$ and $\mathfrak{Hol}_x(M)$ is the *local holonomy algebra*, defined as the Lie algebra of the group

$$\mathrm{Loc\,Hol}_x(M) = \cap \,\mathrm{Hol}_x(U_i)$$

where the intersection has taken over all connected open neighbourhoods $U_i$ of $x$, each endowed with the metric induced from $M$. We have

$$\mathfrak{T}_x \subseteq \mathfrak{Loc\,Hol}_x(M) \subseteq \mathfrak{Hol}_x(M).$$

The following two theorems are basic. The first is according to Nijenhuis [8, 9].

*Theorem 2.1.* Let $(M, g)$ be a connected Riemannian manifold.

(a) For each integer $m$, the set $\{x \in M$ such that $\dim \mathfrak{T}_x \geqslant m\}$ is open.

(b) If $\dim \mathfrak{T}_x$ is independent of $x$, or if $(M, g)$ is real-analytic, then $\mathfrak{T}_x$ is isomorphic to $\mathfrak{Hol}_x(M)$ for all $x \in M$.

See [1] for the term 'real analytic'. In view of (b), we shall say that $M$ has a constant infinitesimal holonomy algebra if $\dim \mathfrak{T}_x$ is independent of $x$.

We shall also need Berger's theorem [1].

*Theorem 2.2.* Let $M$ be a connected, simply connected (but not necessarily complete) $n$-dimensional Riemannian manifold.

(a) If $M$ is irreducible, $\mathrm{Hol}_x(M)$ is isomorphic to one of the following: (i) $SO(n)$, (ii) $U(n/2)$, (iii) $SU(2)Sp(n/4)$, (iv) $SU(n/2)$, (v) $Sp(n/4)$, (vi) $G_2(n = 7)$, (vii) $\mathrm{Spin}(7)(n = 8)$, (viii) $\mathrm{Spin}(9)(n = 16)$, (ix) the isotropy group of a symmetric space of rank $\geqslant 2$.

(b) If $M$ is reducible, $\mathrm{Hol}_x(M)$ is isomorphic to a direct product of groups drawn from the appropriate lower-dimensional versions of the above list, together with the trivial group.

Recall that $M$ is said to be irreducible if $\mathrm{Hol}_x(M)$ acts irreducibly on $T_x(M)$. We shall say that $M$ is *locally* irreducible if the restricted holonomy group $\mathrm{Res}\,\mathrm{Hol}_x(M)$ acts irreducibly, and that it is *strongly* irreducible if $\mathrm{Loc}\,\mathrm{Hol}_x(M)$ acts irreducibly for all $x$. Recall that the restricted holonomy group is obtained by considering parallel transport around contractible loops only. Note that in Berger's theorem, the listed groups must be correctly embedded in $SO(n)$ in each case: see [1] for these embeddings. We shall refer to these lists, and the corresponding list of Lie algebras, as the irreducible (respectively, reducible) Berger lists. Finally, $SO(n)$ and $U(n/2)$ are 'generic', while the other groups in the irreducible list are 'special'. A group in the reducible list is generic if it has at least one generic factor.

It is important to note that Berger's theorem only classifies the holonomy *groups* of (simply connected) manifolds. It is by no means clear that it classifies $\mathfrak{K}_x$ or $\mathfrak{T}_x$. However, the following results, whose proofs we shall merely sketch, are a step in that direction.

*Proposition 2.3.* (a) If the infinitesimal holonomy algebra is constant, then it is classified by the Berger lists.

(b) If $M$ is strongly irreducible, $U$ is a connected open subset of $M$, and the infinitesimal holonomy algebra is constant on $U$, then each $\mathfrak{T}_x$ is classified by the irreducible Berger list for all $x \in U$.

*Proof.* (a) As the restricted holonomy group is isomorphic to the holonomy group of the Riemannian universal cover of $M$, it is classified by the Berger lists. By Nijenhuis' theorem, each infinitesimal holonomy algebra is isomorphic to the algebra of the holonomy group, which of course is also the algebra of the restricted holonomy group.

(b) Given any point $x$ in $M$, it is clear that we can find a connected open neighbourhood $V_x$ such that the holonomy group of $V_x$ coincides with the local holonomy group of $M$ at $x$, and such that

$$\mathrm{Loc}\,\mathrm{Hol}_x(M) = \mathrm{Hol}_x(W_x)$$

for every connected open neighbourhood $W_x$ contained in $V_x$. Given the open set $U$, let $x$ be in $U$ and let $V_x$ be as above. Letting $W_x$ be the intersection of $V_x$ and $U$, we find that the local holonomy group of $U$ coincides with that of $M$ itself, and so $U$ is strongly irreducible. The result now follows as in part (a).

It remains now to consider the consequences of relaxing the condition that the infinitesimal holonomy algebra be constant. A helpful way to think about this is in terms of 'rigidity'. Given a manifold $M$ with a constant infinitesimal holonomy algebra, we know that $\dim \mathfrak{T}_x$ cannot be 'pushed up' at isolated points: according to the first part of Nijenhuis' theorem, this can only be done on open sets. The following proposition will allow us to say more.

*Proposition 2.4.* Let $M$ be a connected Riemannian manifold which is complete or locally irreducible or both. Suppose that $M$ contains no open subset on which the infinitesimal holonomy algebra is constant and generic. Then $M$ has a constant infinitesimal holonomy algebra.

*Proof.* Let $N_0$ be any connected component of the set $\{x \in M \text{ such that} \dim \mathfrak{T}_x = \operatorname*{Max}_{y \in M} \dim \mathfrak{T}_y\}$. By Nijenhuis' theorem, $N_0$ is an open submanifold of $M$. With respect to the induced metric, $N_0$ is (again by Nijenhuis' theorem) a manifold with a constant infinitesimal holonomy algebra, which must, by proposition 2.3, belong to one of the $n$-dimensional Berger lists. By hypothesis, the infinitesimal holonomy algebra is not generic. Hence $\mathfrak{Hol}(N_0)$ is a direct sum of algebras drawn from the *special* Berger lists, and so (since $N_0$ is simply connected) $\mathrm{Hol}(N_0)$ has the form $S_0 \times S_1 \times S_2 \times \cdots$ where each group $S_j$ is special. Here we can take $S_0$ to be the trivial group if necessary; otherwise each $S_j$ acts irreducibly on some subspace $T_x^{(j)}$ of the tangent space $T_x(N_0)$, where $x$ is an arbitrary point in $N_0$. (The action of $S_j$ on $T_x^{(k)}$ is of course trivial if $k \neq j$.) Now since $N_0$ is not complete, we cannot apply the deRham splitting theorem to conclude that $N_0$ is isometric to a product manifold. However, we can still proceed as follows (see [10], p 185). Fix $x$, and let $T_x(N_0) = \Sigma T_x^{(j)}$ as above. Then it is possible to show that for each $j$, there is a submanifold of $N_0$, say $N_0^{(j)}$, passing through $x$, with a tangent space at $x$ which may be identified with $T_x^{(j)}$; and there exists a connected open neighbourhood of $x$, $V_x$, which is isometric to a Riemannian product $V_x^{(0)} \times V_x^{(1)} \times \cdots$ where $V_x^{(j)}$ is an open neighbourhood of $x$ in $N_0^{(j)}$. Hence $\mathrm{Hol}_x(V_x) = \mathrm{Hol}_x(V_x^{(0)}) \times \mathrm{Hol}_x(V_x^{(1)}) \times \cdots$. Now since the infinitesimal holonomy algebra is constant on $N_0$, it is likewise constant on $V_x$, and so Nijenhuis' theorem implies that $\mathfrak{Hol}_x(V_x) = \mathfrak{T}_x$. But $\mathfrak{T}_x = \mathfrak{Hol}_x(N_0)$, and so $\mathrm{Hol}_x(N_0) = \mathrm{Hol}_x(V_x)$ since both $N_0$ and $V_x$ are simply connected. Therefore $S_j = \mathrm{Hol}_x(V_x^{(j)})$, and so with the exception of $V_x^{(0)}$ (which is flat) each $V_x^{(j)}$ is an irreducible manifold with special holonomy. Now irreducible manifolds with special holonomy are Einstein manifolds (see [1]) and so, of course, are flat manifolds. Hence, there is an open neighbourhood of $x$ on which the Ricci tensor is a sum, with constant coefficients, of the metrics on the $V_x^{(i)}$. Therefore, $\nabla \mathrm{Ric} = 0$ at $x$ and hence throughout $N_0$; indeed we have $\nabla \mathrm{Ric} = 0$ on the entire set $\{x \in M \text{ such that} \dim \mathfrak{T}_x = \operatorname*{Max}_{y \in M} \dim \mathfrak{T}_y\}$ and also on its closure.

Let $M_1$ be the complement of that closure, and let $N_1$ be a connected component of the open set $\{x \in M_1 \text{such that } \dim \mathfrak{T}_x = \operatorname*{Max}_{y \in M} \dim \mathfrak{T}_y\}$. Proceeding thus, we find that $\nabla \mathrm{Ric} = 0$ everywhere on $M$. If $M$ is locally irreducible, Schur's lemma now implies that $M$ is an Einstein manifold. The DeTurck–Kazdan [7] theorem now informs us that $M$ is real-analytic—that is, $M$ has an atlas of normal coordinate systems with respect to which the metric is real-analytic. (For a very clear explanation of this theorem, normal coordinates, and their relevance to analyticity, see chapter 5 of [1], in particular p 145.) The Nijenhuis theorem implies that the infinitesimal holonomy algebra of $M$ is constant.

If $M$ is complete rather than locally irreducible, we proceed as follows. Let $\tilde{M}$ be the universal cover of $M$, endowed with the pulled-back metric. Then $\tilde{M}$ is also complete with respect to this metric, as well as being simply connected. Therefore, we can use the deRham splitting theorem [10]: $\tilde{M}$ is globally isometric to the Riemannian product $\tilde{M} = \tilde{M}^{(0)} \times \tilde{M}^{(1)} \times \tilde{M}^{(2)} \times \cdots$ where each $\tilde{M}^{(j)}$ is either flat or irreducible. Now $\nabla\,\mathrm{Ric} = 0$ on $\tilde{M}$, and hence on each $\tilde{M}^{(j)}$. Applying Schur's lemma as before, we find that each $\tilde{M}^{(j)}$ is an Einstein manifold and hence real-analytic. Therefore $\tilde{M}$ is real-analytic and consequently the infinitesimal holonomy algebra of $\tilde{M}$ is constant. The same is therefore true of $M$. This completes the proof.

We shall give two applications of this proposition. The first will be used in the proof of the main theorem 1.1.

*Theorem 2.5 (Rigidity).* Let $M$ be a connected, complete manifold with an infinitesimal holonomy algebra of type $\mathfrak{G}$—that is, $\mathfrak{T}_x \subseteq \mathfrak{G}$ for all $x \in M$, and $\mathfrak{T}_x = \mathfrak{G}$ for at least one $x$. If $\mathfrak{G}$ is isomorphic to the Lie algebra of one of the *special* members of the $n$-dimensional Berger lists, then $\mathfrak{T}_x = \mathfrak{G}$ for all $x \in M$.

*Proof*. Suppose that $M$ contains a connected open set $U$ on which the infinitesimal holonomy algebra is constant and generic. Since it is constant, this algebra is isomorphic to the algebra of one of the members of the $n$-dimensional Berger lists, (proposition 2.3). Let $x \in U$; then $\mathfrak{T}_x \subseteq \mathfrak{G}$. But none of the special members of the Berger lists contain any generic members of either the irreducible or the reducible Berger list. (Recall that the generic members of the reducible list are those with at least one generic factor.) For example, $\mathrm{Sp}(k)$ contains no members of the $4k$-dimensional irreducible Berger list other than itself, and it contains no members of the $4k$-dimensional reducible Berger list other than those of the form $\mathrm{Sp}(k') \times \mathrm{Sp}(k'')$, where $k' + k'' \leqslant k$, and none of these are generic. (Bear in mind our remarks concerning 'correct' embeddings, after theorem 2.2.) The contradiction implies that $M$ satisfies the conditions of proposition 2.4, and so the infinitesimal holonomy algebra is constant. Since it is isomorphic to $\mathfrak{G}$ at one point, $\mathfrak{T}_x = \mathfrak{G}$ for all $x$, and this completes the proof.

A connected, complete Riemannian manifold with a constant special infinitesimal holonomy algebra is therefore 'rigid' in the following sense. $\mathfrak{T}_x$ cannot be 'pushed down' to a sub-algebra at a point or even on a proper open subset, as long as the infinitesimal holonomy algebra is of type $\mathfrak{G}$. Intuitively, the manifold 'resists being flattened'; if $\mathfrak{T}_x$ is 'pushed down' at $x$, then $\mathfrak{T}_y$ will automatically 'push up' to a generic algebra at some other point $y \in M$ (and therefore on an open neighbourhood of $y$). This remarkable interaction of the infinitesimal holonomy algebras at various points is ultimately due to the real analyticity of Einstein metrics.

As another application of proposition 2.4, and to conclude this section, we analyse the structure of connected Riemannian manifolds with *non-constant* infinitesimal holonomy algebras. It is convenient at this point to assume that $M$ is strongly irreducible. Then according to part (b) of proposition 2.3, if $U$ is a connected open subset on which $\mathfrak{T}_x$ is constant, then this algebra (denoted $\mathfrak{T}(U)$) is isomorphic to the algebra of one of the members of the irreducible Berger list. Furthermore, if $x \in \partial U$, the boundary of $U$, then $\mathfrak{T}_x \subseteq \mathfrak{T}(U)$. Now let us consider the contrary; then $R_x$, $(\nabla R)_x$ and so on do not satisfy the algebraic relations which define $\mathfrak{T}(U)$. But if that is the case, then evidently we can find an open neighbourhood of $x$ on which these relations are not satisfied, and this contradicts the fact that $x$ is in the boundary.

Now by proposition 2.4 (recall that $M$ is locally irreducible if it is strongly irreducible) $M$ must contain a non-empty open subset on which the infinitesimal holonomy algebra is constant and generic. Let $M_g$ be the union of all such; we call $M_g$ the *generic submanifold*

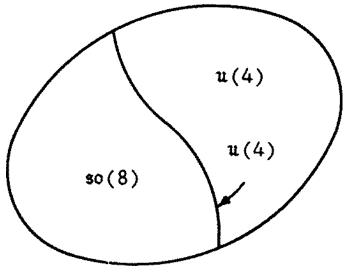


**Figure 1.** A manifold of holonomy SO(8).

of $M$. If $x$ is any point of $M$ such that $\mathfrak{T}_x$ is generic, then $x$ has an open neighbourhood $V_x$ on which the infinitesimal holonomy algebra is generic. Now either the infinitesimal holonomy algebra is constant on $V_x$ (in which case $x \in M_g$) or it is not; in the latter case, $V_x$ must contain a non-empty open subset on which the infinitesimal holonomy algebra is constant and generic, and so either $x \in M_g$ or $x \in \partial M_g$. Hence every 'generic point' is in $M_g$ or its boundary.

Clearly $\mathfrak{T}_x$ is isomorphic either to $\mathfrak{SO}(n)$ or to $\mathfrak{U}(\frac{1}{2}n)$ for all $x \in M_g$, and the subset of $M_g$ on which $\mathfrak{T}_x = \mathfrak{SO}(n)$ is open (possibly empty). If there is a non-empty open subset on which $\mathfrak{T}_x = \mathfrak{U}(\frac{1}{2}n)$, then $\mathfrak{T}_x = \mathfrak{U}(\frac{1}{2}n)$ on the boundary. Figure 1 gives a schematic indication of one possible situation in the case of an eight-dimensional manifold. Notice that we have taken $\mathfrak{T}_x = \mathfrak{U}(4)$ on the boundary of $M_g$. There are other possibilities, but it would not be possible to have $\mathfrak{T}_x = \mathfrak{SO}(8)$ or $\mathfrak{spin}(7)$ on the boundary, as these are not contained in $\mathfrak{U}(4)$. Beyond this, our results so far do not permit us to be more specific as to the behaviour of $\mathfrak{T}_x$ on the boundary of $M_g$. Indeed, we have not even shown that, in this case, $\mathfrak{T}_x$ must belong to *either* of the Berger lists. In the next section, it will be proved that $\mathfrak{T}_x$ is in fact classified by the Berger lists for *all* $x \in M$ (theorem 3.1). However, theorem 3.1 does *not* imply that when $M$ is irreducible—or even strongly irreducible—every $\mathfrak{T}_x$ must belong to the *irreducible* Berger list. Therefore, even if $M$ is strongly irreducible, it is possible to find that $\mathfrak{T}_x$ is a member of the reducible Berger list when $x$ is in the boundary of $M_g$. It is possible, for example, that $\mathfrak{T}_x$ could be isomorphic to $\mathfrak{SO}(n)$ everywhere except at one point, where it is the zero algebra, despite the fact that the zero algebra belongs to the reducible Berger list. (It corresponds, of course, to the trivial group.) Choosing normal coordinates at this point, we will find that all partial derivatives of the metric vanish there; this underlines the fact that we are dealing with metrics which are not analytic.

In the complement of the closure of $M_g$, the behaviour of the infinitesimal holonomy algebra is more strongly constrained. Let $N$ be a connected component of this complement (assuming that it is not empty). Then $N$ is locally irreducible (since $M$ is strongly irreducible) and $\mathfrak{T}_x$ cannot be generic for any $x \in N$. Hence proposition 2.4 implies that the infinitesimal holonomy algebra is constant on $N$; and by proposition 2.3, it is isomorphic to one of the special members of the $n$-dimensional irreducible Berger list. We see, then, that $\mathfrak{T}_x$ is classified by the irreducible Berger lists everywhere, except possibly on the boundary of the generic submanifold. Figure 2 represents a simple example in which the complement of the closure of $M_g$ is non-empty. More generally, it would be possible to have more than one region on which $\mathfrak{T}_x$ is special, but these regions must be mutually disconnected if their infinitesimal holonomy algebras differ.

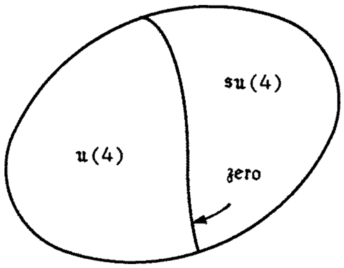


**Figure 2.** A manifold of holonomy U(4).

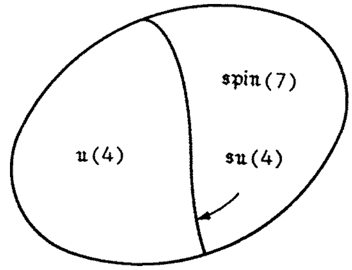


**Figure 3.** A manifold of holonomy SO(8).

Figure 3 represents an interesting case which may be possible in eight dimensions. Since neither $U(4)$ nor Spin(7) is contained in the other, we cannot have either $\mathfrak{T}_x = \mathfrak{U}(4)$ or $\mathfrak{spin}(7)$ on the boundary. On the other hand, $SU(4)(=\text{Spin}(6))$ is contained in both $U(4)$ and Spin(7), and so $\mathfrak{SU}(4)$ is possible, as are smaller algebras such as $\mathfrak{Sp}(2)$ (=$\mathfrak{spin}(5)$) or even the zero algebra. It would be most interesting to have an explicit example of a compact manifold of this kind, particularly if the size of the generic region could be varied.

## 3. The curvature algebra

The infinitesimal holonomy algebra provides a link between local geometric data and the holonomy group of a connected Riemannian manifold. However, theorem 1.1 is stated in terms of a simpler object, the curvature algebra $\mathfrak{K}_x$. At this point it is by no means clear that the curvature algebras are classified by the Berger lists; indeed, we have yet to demonstrate this completely even for $\mathfrak{T}_x$, except for manifolds with constant infinitesimal holonomy algebras. Clearly we need a complete classification of both $\mathfrak{K}_x$ and $\mathfrak{T}_x$. This can be done by means of the simple device of constructing a Riemannian manifold which has the same curvature (at a given point) as $M$, but which is analytic in normal coordinates.

*Theorem 3.1.* Let $M$ be any $n$-dimensional Riemannian manifold, and let $x \in M$. Then $\mathfrak{K}_x$ is isomorphic to the Lie algebra of some member of one of the $n$-dimensional Berger lists; and similarly for $\mathfrak{T}_x$.

*Proof.* Let $S$ denote the $(0, 4)$ version of the curvature tensor at $x$. If $T_x(M)$ is regarded as a manifold, then each tangent space of $T_x(M)$ can be identified with $T_x(M)$ in a natural way, and this identification will be understood henceforth. Let $g(x)$ denote the metric tensor of $M$ evaluated at $x$. We define a $(0, 2)$ tensor, $h$, on $T_x(M)$ as follows. Let $y \in T_x(M)$ and let $X, Y$ be tangent vectors to $T_x(M)$ at $y$. Then set

$$h(y)(X, Y) = g(x)(X, Y) + \tfrac{1}{3}S(X, y, y, Y).$$

Notice that $S(Y, y, y, X) = S(y, X, Y, y) = S(X, y, y, Y)$ so that $h$ is a symmetric tensor field on $T_x(M)$. Let $N$ be a connected open neighbourhood of the origin of $T_x(M)$ such that $h(y)$ is positive—definite for all $y \in N$; such an $N$ clearly exists, since $g(x)$ is positive—definite. If $g^N$ denotes the restriction of $h$ to $N$, then $(N, g^N)$ is a $n$-dimensional Riemannian manifold. With respect to the global coordinate system induced on $T_x(M)$ by an orthonormal basis of $T_x(M)$, we have $g^N_{ij}(y) = \delta_{ij} + \frac{1}{3}\sum_k \sum_l S_{iklj} y^k y^l$, where the $S_{iklj}$ are the components of $S$ with respect to the given basis, and the $y^j$ are the coordinates of the point $y$ (that is, the components of the vector $y$). Evidently $g^N_{ij}(0) = \delta_{ij}$ and $(\partial_k g^N_{ij})(0) = 0$, and so these coordinates are normal at the origin for $g^N$. Clearly $g^N$ is a real-analytic in normal coordinates.

Now let $(P, g)$ be *any* Riemannian manifold such that the metric is real-analytic in normal coordinates. Then [11] the components of $g$ with respect to normal coordinates may be expressed as a power series of the form

$$\begin{aligned} g_{ij}(x) = \delta_{ij} + \tfrac{1}{3}\sum_k\sum_l R_{iklj}(0)x^k x^l + \tfrac{1}{6}\sum_k\sum_l\sum_m(\nabla_k R_{ilmj})(0)x^k x^l x^m + \cdots \\ + C_r \sum_k\sum_l \ldots [(\nabla_k\nabla_l \ldots R_{impj})(0) + \text{terms involving lower order} \\ \text{covariant derivatives of } R]x^k x^l \cdots + \cdots \end{aligned}$$

where $R_{iklj}(0), (\nabla_k R_{ilmj})(0)$ etc denote the components of $R$, $\nabla R$, etc evaluated at the origin of coordinates, and where $C_r$ (the coefficient of the $r$th order term) is some universal constant. The precise form of this formula is very complicated (see [11]) and not necessary for our purposes. Recall that, to define the curvature algebra and the infinitesimal holonomy algebra, we think of the curvature and its covariant derivatives as endomorphisms of the tangent space at any point: in simpler language, we 'feed' tangent vectors to these tensors until precisely one contravariant and one covariant index remain free. This allows us to think of the curvature tensor and its covariant derivatives as matrices, which generate the various algebras by ordinary matrix multiplication. Now suppose that, in the above expansion, every term beyond the quadratic vanishes. Then, leaving aside technicalities to be dealt with below, the higher covariant derivatives of the curvature tensor either vanish or can be expressed as matrix products of the lower ones. (This is what we mean by 'terms involving lower order covariant derivatives of $R$'.) The upshot is that the infinitesimal holonomy algebras of such a metric coincide with the curvature algebras. Intuitively, this is easy to understand: the infinitesimal holonomy algebra differs from the curvature algebra only because the former depends on all of the terms in the Taylor expansion of the metric. If, as in the case of the metric on $N$, the expansion stops at the quadratic level, then we expect to obtain nothing new from the higher derivatives.

Now since $g^N$ is analytic in normal coordinates, we can compute its curvature $R^N$ and the derivatives of the curvature by simply equating coefficients in the above power series—bearing in mind, of course, that the coefficients of $x^k x^l x^m \ldots$ will automatically be symmetrical in the indices $k, l, m \ldots$. Thus we obtain

$$\begin{aligned} &R^N_{iklj}(0) + R^N_{ilkj}(0) = S_{iklj} + S_{ilkj} \\ &(\nabla^N_k R^N_{ilmj})(0) + \mathrm{Sym}(k, l, m) = 0 \\ &[(\nabla^N_k\nabla^N_l \ldots R^N_{impj})(0) + \text{terms involving lower order covariant derivatives of } R^N] \\ &\qquad + \mathrm{Sym}(k, l, \ldots, m, p) = 0 \end{aligned}$$

where Sym( ) denotes additional terms such that the resulting expression is symmetrical in the indicated indices and 0 denotes $0 \in T_x(M)$. However, both $R^N(0)$ and $S$ are the curvature of certain metrics, and so both satisfy the usual identities. The effect of these identities is that the Sym( ) terms can be dropped from the above relations. For example, the first equation implies that $S_{iklj} + S_{ilkj} - S_{kilj} - S_{klij} - S_{ikjl} - S_{ijkl} + S_{ljik} + S_{lijk} =$ {a similar expression in$R^N_{iklj}(0)$}.

The symmetries of the curvature tensor reduce the left-hand side to $4S_{iklj} + 2S_{ilkj} + 2S_{ijlk}$ which, by the first Bianchi identity, is just $6S_{iklj}$. Therefore we have $R^N_{iklj}(0) = S_{iklj}$. In the same way, the differential identities satisfied by curvature tensors reduce the second equation to $(\nabla^N_k R^N_{ilmj})(0) = 0$ (only at the origin of $T_x(M)$, of course), and so on. In general, the higher derivatives of $R^N$, evaluated at 0, can be expressed algebraically in terms of the lower-order derivatives, also evaluated at 0. If we denote the curvature algebra of $N$ at the origin by $\mathfrak{K}_0(N)$, and the infinitesimal holonomy algebra by $\mathfrak{T}_0(N)$, then we have $\mathfrak{T}_0(N) = \mathfrak{K}_0(N)$. (We are not asserting—nor is it true in general—that $\mathfrak{T}_x(M)$ is isomorphic to $\mathfrak{K}_x(M)$.)

The fact that $g^N$ is analytic in normal coordinates allows us to invoke Nijenhuis' theorem, and so we have $\mathfrak{T}_0(N) = \mathfrak{hol}_0(N)$. The latter must be isomorphic to the Lie algebra of some member of one of the Berger lists. (Recall that Berger's theorem does not require completeness.) But the equation $R^N_{iklj}(0) = S_{iklj}$ implies that $\mathfrak{K}_0(N) = \mathfrak{K}_x(M)$, since $g^N_{ij}(0) = \delta_{ij}$ and the $S_{iklj}$ are components with respect to an orthonormal basis of $T_x(M)$. Therefore, we have $\mathfrak{K}_x(M) = \mathfrak{K}_0(N) = \mathfrak{T}_0(N) = \mathfrak{hol}_0(N)$, and so $\mathfrak{K}_x(M)$ belongs to one of the Berger lists.

Let $S_{miklj}$ denote the components of $(\nabla R)_x$ with respect to the same orthonormal basis of $T_x(M)$ as was used above. Then using the coordinates on $T_x(M)$ corresponding to this basis, we modify the definition of $g^N$ as follows:

$$g^N_{ij}(y) = \delta_{ij} + \tfrac{1}{3}\sum_k\sum_l S_{iklj}\,y^k y^l + \tfrac{1}{6}\sum_m\sum_k\sum_l S_{miklj}\,y^m y^k y^l$$

where $N \subseteq T_x(M)$ is defined in the obvious way. Proceeding as before, we obtain $S_{iklj} = R^N_{iklj}(0)$ and $S_{miklj} = (\nabla^N_m R^N_{iklj})(0)$, while every higher derivative can be expressed in terms of these two. Thus, we have $\mathfrak{K}^{(1)}_0(N) = \mathfrak{T}_0(N)$, and the latter is again isomorphic to $\mathfrak{hol}_0(N)$ by Nijenhuis' theorem. Since $\mathfrak{K}^{(1)}_x(M)$ is clearly isomorphic to $\mathfrak{K}^{(1)}_0(N)$, we again find that $\mathfrak{K}^{(1)}_x(M)$ is classified by Berger's lists. In fact, the same is evidently true of $\mathfrak{K}^{(j)}_x(M)$ for every $j$. But $\mathfrak{T}_x(M) = \mathfrak{K}^{(j)}_x(M)$ for some sufficiently large $j$, and so $\mathfrak{T}_x(M)$ is also classified by Berger's lists. This completes the proof.

Notice that the holonomy structure of $M$ plays no role in the proof, and so there is no reason to expect that $N$ will be irreducible if $M$ is irreducible—or even strongly irreducible. For example, the curvature of $M$ could vanish at $x$, in which case $\mathrm{Hol}_0(N)$ is trivial. Thus, we cannot prove that $\mathfrak{K}_x(M)$ or $\mathfrak{T}_x(M)$ must belong to the irreducible Berger list for all points in an irreducible or strongly irreducible Riemannian manifold. In order to obtain such results, one needs an even stronger condition. For example, let $\mathrm{Inf\,Hol}_x(M)$ denote the connected subgroup of $\mathrm{Loc\,Hol}_x(M)$ corresponding to $\mathfrak{T}_x(M)$. If the action of $\mathrm{Inf\,Hol}_x(M)$ on $T_x(M)$ is irreducible, then indeed $\mathfrak{T}_x$ must belong to the irreducible Berger list.

## 4. The proof of theorem 1.1

We begin by observing that, according to theorem 3.1, $\mathfrak{K}_x$ is an algebra in one of the Berger lists, and hence the same is true of $\mathfrak{G}$. For the sake of concreteness, let us take $\mathfrak{G}$ to be the symplectic algebra $\mathfrak{Sp}(n/4)$; the reader will find it easy to make the necessary technical modifications for the other special Berger algebras. The statement that $\mathfrak{K}_y \subseteq \mathfrak{Sp}(n/4)$ has the following interpretation: the tangent space $T_y(M)$ must admit a pair of endomorphisms $J$, $K$, satisfying $J^2 = K^2 = -1$, $J \circ K = -K \circ J$, and $R_y \circ J = J \circ R_y$, $R_y \circ K = K \circ R_y$. These last equations have an analytic as well as an algebraic significance: they are the local integrability conditions for the equations $\nabla J = 0$, $\nabla K = 0$. Hence we can arrange for these equations to hold on a suitable small neighbourhood of $y$, and so we have $(\nabla R)_y \circ J = J \circ (\nabla R)_y$, and similarly for the higher derivatives. So in fact the infinitesimal holonomy algebras satisfy $\mathfrak{T}_y \subseteq \mathfrak{Sp}(n/4)$ for all $y \in M$. Since $\mathfrak{K}_x = \mathfrak{Sp}(n/4)$ by hypothesis, and $\mathfrak{K}_x \subseteq \mathfrak{T}_x$ always, we have $\mathfrak{T}_x = \mathfrak{Sp}(n/4)$. Using theorem 2.5, we now have $\mathfrak{T}_y = \mathfrak{Sp}(n/4)$ for all $y$, and now Nijenhuis' theorem gives us the desired result. This completes the proof.

## 5. Conclusion

The overall plan of this work may be summarized as follows. We wish to 'obtain holonomy from curvature', and Nijenhuis' theorem opens the way to this; however, there are two main obstacles in our path. The first is that Nijenhuis' theorem requires information on the infinitesimal holonomy algebras, but the curvature tensor itself only yields the curvature algebras. We deal with this, in the proof of theorem 3.1, by relating the metric to a much simpler one for which the infinitesimal holonomy algebra is (by construction) isomorphic to the curvature algebra, on a certain neighbourhood $N$. This is the relation $\mathfrak{T}_0(N) = \mathfrak{K}_0(N)$. The second obstacle is that Nijenhuis' theorem requires analyticity. This is handled by noting that our information on the curvature algebras allows us to force our manifolds to be Einstein manifolds, which in turn allows us to use the Kazdan–DeTurck analyticity theorem.

From a practical point of view, our results reduce the determination of the holonomy group to a computation of curvature algebras (leaving aside the topological questions mentioned in the introduction). The structure constants of the curvature algebra at some point can be obtained as follows. Choose an orthonormal basis at that point, and regard the curvature tensor components, $R_{ijkm}$, as a set of antisymmetric matrices labelled by $k$ and $m$. Because of the antisymmetry in $k$ and $m$, one obtains $n(n-1)/2$ matrices in this way. These matrices will not, however, be linearly independent (unless the curvature algebra is generic, so that these matrices span the whole orthogonal algebra); instead, they span some subspace of the orthogonal algebra. In principle it is now straightforward linear algebra to identify a basis of this subspace, and the structure constants are then evaluated by computing ordinary matrix commutators. An alternative approach, if one suspects that the curvature algebra is isomorphic to some given sub-algebra of the orthogonal algebra (guided in one's guesses by theorem 3.1, of course) is to check that the above matrices satisfy the simple linear relations which define the Berger algebras. (These are given in chapter 10 of [1], for example.) In practice, both approaches might well involve heavy computations, but it should not be difficult to implement them on a computer algebra program such as MathTensor [12], which automatically produces the curvature components when presented with the components of a metric.

This work was motivated by the observation, in [4], that the holonomy algebra of a Riemannian manifold is not necessarily isomorphic to any of its curvature algebras. Gibbons *et al* solve this problem by constructing a parallel spinor on $M$, which forces the spin holonomy to be special. Although theorem 1.1 shows that this additional labour is unnecessary, the technique used by Gibbons *et al* is of independent interest. The relationship between holonomy, curvature, and parallel spinors is particularly subtle when the manifold is topologically non-trivial, and will be the subject of a later report.

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